

The generalized Buckley-Leverett System. Solvability

Nikolai Chemetov¹, Wladimir Neves²

Abstract

We propose a new approach to the mathematical modeling of the Buckley- Leverett system, which describes two-phase flows in porous media. Considering the initial-boundary value problem for a deduced model, we prove the solvability of the problem. The solvability result relies mostly on the kinetic method.

Contents

1	Introduction	2
2	Non-linear porous-media theory	3
3	Functional notation and background	7
4	Statement of the Stokes B-L system	7
5	Existence of weak solution	10
5.1	Parabolic approximation	10
5.2	The limit transition on $\varepsilon \rightarrow 0$	10
5.2.1	The main idea of the limit transition. Sketch of the proof	11
5.2.2	Proof of Proposition 5.1	12
5.2.3	Rigorous proof of the limit transition	13
6	Statement of the quasi-stationary Stokes B-L system	20
6.1	Existence of weak solution. The limit transition on $\tau \rightarrow 0$	21

¹Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal. E-mail: chemetov@ptmat.fc.ul.pt.

²Institute of Mathematics, Federal University of Rio de Janeiro, C.P. 68530, Cidade Universitária, 21945-970, Rio de Janeiro, Brazil. E-mail: wladimir@im.ufrj.br.

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1 Introduction

We propose a new and more general approach to the mathematical study of the Buckley-Leverett system, which describes two-phase flows in porous media, that is, the simultaneous motion of two immiscible incompressible liquids (e.g. water and oil) in a porous medium. This study is of practical interest in connection with the planning and operation of oil wells and also brings some challenging mathematical questions. Indeed, the mathematical standard model of the Buckley-Leverett system is given by a scalar multidimensional conservation law, which describes the evolution of the saturation according to the seepage velocity field, where this field is given by the Darcy's Law (empirical) equation, i.e.

$$\text{seepage velocity} = \text{equivalent mobility}(\text{saturation}) \times \text{pressure gradient}.$$

Since the equivalent mobility of the porous medium is a function of the saturation, which should be a bounded and measurable function, this brings an enormous mathematical difficult. We have to solve a scalar conservation laws in the class of roughly coefficients. Even nowadays with the best results of Panov for scalar conservation laws with discontinuous flux published in ARMA, see [23], we are not allowed to solve the Buckley-Leverett system in this way. In order to pass the above difficult, the Buckley-Leverett system has been significantly simplified in many works, for instance see: Córdoba, Gancedo, Orive [6], Frid [10], Perepelitsa, Shelukhin [24]. In the article of Luckhaus, Plotnikov [19] has considered a stationary case of the Buckley-Leverett system. Many authors have proposed interesting ideas, but most of them focused on the saturation equation. Mainly reducing the Buckley-Leverett system to a (non)degenerate elliptic-parabolic partial differential system, here we address some of the important works on this subject: Antontsev, Kazikhov, Monakhov [1], Arbogast [2], Chen [5], Lenzinger, Schweizer [16], Sazhenkov [27] and further references cited therein.

In the present work we change the focus and put more attention to the equation of velocity. So we propose a generalized Darcy's law equation, which is no physically longer than the standard one. This new formulation bring to us enough regularity of the seepage velocity field in order to obtain solvability of the system using the nice idea of Kinetic Theory. This is the most part of the motivation to introduce a general approach to the mathematical study of the Buckley-Leverett system. In the rest of the introduction, we give a general presentation of the generalized Darcy's law equation in a homogeneous and isotropic medium for one phase flow.

The theory of flows in porous media has also a number of similarities with several other processes occurring in the continuum physics as, for instance, problems of infiltration, displacement of electricity through dielectric media, heat transfer, etc. Indeed, after suitable averaging the porous media and the liquids filling them can be regarded as continuous medium under natural assumptions made about the pore system, see Scheidegger [28, 29]. Analogously to the resistance for the conductors of electricity, we have here the porosity as a characteristic of the porous media.

One observes that, for very short time scales or high frequency oscillations, a time derivative of flux may be added to Darcy's law, which results in valid solutions at very small times. In heat transfer, a similar idea is called the Cattaneo's law which is a modified version of the standard Fourier one, hence in analogy we have the following equation for the velocity field \mathbf{v} ,

$$\tau \partial_t \mathbf{v} + \frac{\mu}{\kappa} \mathbf{v} = -\nabla p, \quad (1.1)$$

where p is the pressure, μ is the dynamic viscosity, κ is the permeability and τ is a very small time constant. The parameter τ causes this equation to reduce to the normal form of Darcy's law at usual times. The main reason for doing this is that the regular groundwater flow equation (diffusion equation) leads to singularities at constant head boundaries at very small times. Analogously to the heat transfer case, this form is more mathematically rigorous, which leads to a hyperbolic groundwater flow equation.

Another extension to the traditional form of Darcy's law is Brinkman's term (introduced in 1947 by Brinkman), which is used to account for transitional flow between boundaries,

$$\nu \Delta \mathbf{v} + \frac{\mu}{\kappa} \mathbf{v} = -\nabla p, \quad (1.2)$$

where ν is an effective viscosity term. This correction term accounts for flow through medium where the grains of the media are porous themselves. In this paper, we are going to consider both extensions (1.1) and (1.2) of Darcy's law equation for positive viscosity ν and non-negative parameter τ . The combination of (1.1) and (1.2) is sometimes called Brinkman-Forchheimer equation in a porous media literature, see for instance [30], [31] and [33].

It is important to observe that, such generalized Darcy's law models are described also considering the homogenization theory, see for instance [12], and we address further [31].

2 Non-linear porous-media theory

In this section we are going to formulate the porous-media theory for two immiscible incompressible liquids in a porous medium. Here, as it is standard in the formalism of continuum mechanics, we have enough regularity, integrability, etc. of the involved functions to proceed any mathematical computation. Moreover, we assume some simplifications and analogies to obtain our model, but taken out any un-physical considerations.

Let $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ ($d = 1, 2, 3$) be the points in the time-space domain. First, let us consider the porosity $m(\mathbf{x})$, which is the proportion of the pore volume in an infinitesimal part of the porous medium containing the point $\mathbf{x} \in \Omega$.

One could describes the problem of two-phase flow in a porous medium with the two main elements $\mathbf{v}_i(t, \mathbf{x})$, $s_i(t, \mathbf{x})$ ($i = 1, 2$) respectively the velocity field of each fluid, which takes value in \mathbb{R}^d , and the saturation of each component,

which is a scalar function. In fact, the saturation of each component represents the local proportion of the pore space occupied by the i^{th} -phase, thus we must have $0 \leq s_i \leq 1$ ($i = 1, 2$) and

$$s_1 + s_2 = 1. \quad (2.3)$$

Moreover, the velocity field of each component is obtained from an average of the flow rate of the i^{th} -phase divide by an unitary area, used called seepage velocity. Related to the speed of the velocities, we are not going to consider the non-linear convection terms.

Now, we are in position to present the equations concerned the immiscible incompressible multiple phase flow. At this point, we follow reference [29] Part IX, and address also [28]. The evolution of the saturation is driven by the velocity field described by the following continuity equation

$$m \partial_t(\rho_i s_i) + \text{div}(\rho_i \mathbf{v}_i) = 0, \quad (i = 1, 2) \quad (2.4)$$

where ρ_i is mass density of the i^{th} -phase of the porous medium and the velocity field satisfies the generalized Darcy's law equation

$$\tau_i \rho_i \partial_t \mathbf{v}_i - \nu_i \Delta \mathbf{v}_i + \frac{\mu_i}{k_0 k_{ri}(s_1)} \mathbf{v}_i = -\nabla p_i + \rho_i g h, \quad (i = 1, 2) \quad (2.5)$$

where for each component $i = 1, 2$, $p_i(t, \mathbf{x})$ is the pressure, ν_i , μ_i respectively the viscosity and dynamic viscosity, k_{ri} is the relative permeability and τ_i is the time-delay parameter. Moreover, $k_0(\mathbf{x})$ is the absolute permeability of the porous medium and $\rho g h$ is the external gravitational force, which could be dropped, since we are considering an horizontal domain (reservoir) whose height is negligible compared to the other dimensions. As it stands, the equations (2.4) and (2.5) form a system of four partial differential equations, where the unknowns are the velocity and saturation of each component. The pressure is obtained a posteriori by the velocity field, exactly as a Lagrangian multiplier in the Leray's theory for incompressible Navier-Stokes equations.

As it is standard in porous media theory, in order to simplify the model, we assume that $m = k_0 = \rho_1 = \rho_2 \equiv 1$, and further

$$\tau_1 = \tau_2 =: \tau, \quad \nu_1 = \nu_2 =: \nu.$$

In fact, the time-delay τ is a nonnegative very-small parameter, and here, we are going to consider two cases, that is $\tau > 0$ and $\tau = 0$. The viscosity positive parameter ν is also very small, i.e. $0 < \nu \ll 1$. Moreover, by the Laplace's formula (experimental one), it follows that

$$p_1(t, \mathbf{x}) - p_2(t, \mathbf{x}) = p_c(\mathbf{x}, s_1),$$

where p_c is the capillarity pressure, and by the Buckley-Leverett assumptions, we suppose that $p_c \equiv 0$, thus we have $p_1 = p_2 =: p$. Then, from (2.4) and (2.5) we obtain respectively

$$\partial_t s_i + \text{div}(\mathbf{v}_i) = 0 \quad (2.6)$$

and

$$\tau \partial_t \mathbf{v}_i - \nu \Delta_x \mathbf{v}_i + \lambda_i \mathbf{v}_i = -\nabla_x p, \quad (2.7)$$

where $\lambda_i = \mu_i/k_{ri}$ ($i = 1, 2$).

Now, we are going to proceed in order to obtain the final model, which is written in terms of the (total) velocity $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$. We derive it assuming temporarily the one-dimensional case and denote $F(t, x) = -\partial_x p(t, x)$ for simplicity. First, let us understand precisely the rule of the parameter τ in the generalized Darcy's law equation. Set $t = \tau \hat{t}$ and define

$$\mathbf{v}(t, x) =: \hat{\mathbf{v}}(\hat{t}, x), \quad F(t, x) =: \hat{F}(\hat{t}, x),$$

where we have dropped the under-script ($i = 1, 2$) in order to simplify the notation. Therefore, we have for very short times, i.e. $t = O(\tau)$, that the velocity field $\mathbf{v}(t, x)$ behaves like in "normal" times, since $\mathbf{v}(t, x) = \hat{\mathbf{v}}(O(1), x)$, where $\hat{\mathbf{v}}$ satisfies the equation

$$\partial_{\hat{t}} \hat{\mathbf{v}} - \nu \partial_x^2 \hat{\mathbf{v}} + \lambda \hat{\mathbf{v}} = \hat{F}.$$

On the other hand, for $t = O(1)$, $\mathbf{v}(t, x)$ behaves like in permanent regime, since $\mathbf{v}(t, x) = \hat{\mathbf{v}}(O(1/\tau), x)$. Consequently, for each $\delta > 0$ we are allowed to suppose that $\mathbf{v}(t + \delta, x) \simeq \mathbf{v}(t, x)$ for usual times $t = O(1)$. Further, we apply formally the Faedo-Galerkin's method to equation (2.7), i.e., we consider

$$\mathbf{v}(t, x) = \sum_{n=-\infty}^{\infty} \mathbf{v}_n(t) \exp(inx) \quad \text{and} \quad F(t, x) = \sum_{n=-\infty}^{\infty} c_n(t) \exp(inx).$$

Then, from (2.7) we obtain for each n

$$\tau \mathbf{v}'_n(t) + \Lambda_n \mathbf{v}_n(t) = c_n(t),$$

where $\Lambda_n := \lambda + n^2 \nu$. Let $\delta > 0$ be sufficiently small, we resolve the above differential equation from t to $t + \delta$ for some usual time $0 < t = O(1)$, that is

$$\mathbf{v}_n(t) \left(e^{(\Lambda_n/\tau)(t+\delta)} - e^{(\Lambda_n/\tau)t} \right) = \frac{1}{\tau} \int_t^{t+\delta} c_n(\xi) e^{(\Lambda_n/\tau)\xi} d\xi,$$

where we have used $\mathbf{v}_n(t + \delta) = \mathbf{v}_n(t)$. Hence dividing by δ and taking the limit as $\delta \rightarrow 0^+$, we have

$$\mathbf{v}_n(t) = \frac{1}{\Lambda_n} c_n(t).$$

Therefore, it follows that

$$\begin{aligned} \mathbf{v}(t, x) &= \sum_{n=-\infty}^{\infty} \frac{1}{\Lambda_n} c_n(t) \exp(inx) \\ &= \frac{1}{\Lambda} F(t, x), \end{aligned}$$

where $1/\Lambda$ is the value of the absolutely convergent series $\sum 1/(\lambda + n^2\nu)$. From the above expression, i.e. $\Lambda_i \mathbf{v}_i = F$ ($i = 1, 2$), we obtain

$$\mathbf{v} := \mathbf{v}_1 + \mathbf{v}_2 = \left(\frac{\Lambda_1 + \Lambda_2}{\Lambda_1 \Lambda_2} \right) F(t, x),$$

or also denoting

$$\Lambda_{eq} := \frac{\Lambda_1 \Lambda_2}{\Lambda_1 + \Lambda_2},$$

we have

$$F = \Lambda_{eq} \mathbf{v} = \Lambda_1 \mathbf{v}_1 = \Lambda_2 \mathbf{v}_2,$$

that is,

$$\mathbf{v}_i(t, x) = \frac{\Lambda_{eq}}{\Lambda_i} \mathbf{v}(t, x) \quad (i = 1, 2).$$

Finally, taking as motivation the above formulation, we derive our porous-media generalized model for two immiscible incompressible liquids in a porous media in the following way. From equation (2.6) written for each component and added, we obtain

$$\operatorname{div} \mathbf{v} = 0. \quad (2.8)$$

Moreover, denoting $u := s_1$, we have

$$\partial_t u + \mathbf{v} \cdot \nabla g(u) = 0, \quad (2.9)$$

where $g(u) := \Lambda_{eq}/\Lambda_1$. Finally, taking in account the parabolic/elliptic equation (2.7), we have

$$\tau \partial_t \mathbf{v}_1 - \nu \Delta \mathbf{v}_1 + \lambda_1 \mathbf{v}_1 = -\nabla p,$$

and

$$\tau \partial_t \mathbf{v}_2 - \nu \Delta \mathbf{v}_2 + \lambda_2 \mathbf{v}_2 = -\nabla p,$$

hence adding these two equations, we get

$$\tau \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + h(u) \mathbf{v} = -2\nabla p, \quad (2.10)$$

where

$$h(u) = \left(\frac{\lambda_1}{\Lambda_1} + \frac{\lambda_2}{\Lambda_2} \right) \Lambda_{eq} = \frac{\lambda_1 \Lambda_2 + \lambda_2 \Lambda_1}{(\Lambda_1 + \Lambda_2)}.$$

The above deduced model, that is, equations (2.8), (2.9) and (2.10) will be called **Stokes-Buckley-Leverett system** (or for brevity **Stokes B-L system**) for $\tau \neq 0$. Moreover, when $\tau = 0$, we are going to say **quasi-stationary Stokes B-L system**.

3 Functional notation and background

Let $T > 0$ be any fixed real number and $\Omega \subset \mathbb{R}^d$ (with $d = 1, 2$ or 3) is an open and bounded domain having a C^2 - smooth boundary Γ . We define by

$$\Omega_T := \Omega \times (0, T), \quad \Gamma_T := \Gamma \times (0, T).$$

The outside normal to Ω at $\mathbf{x} \in \Gamma$ is denoted by $\mathbf{n} = \mathbf{n}(\mathbf{x})$.

In the paper we will use the standard notations for the Lebesgue function space $L^p(\Omega)$ and the Sobolev spaces $W^{s,p}(\Omega)$ and $H^s(\Omega) \equiv W^{s,2}(\Omega)$ where a real $p \geq 1$ is the integrability indice and a real $s \geq 0$ is the smoothness indice. The vector counterparts of these spaces are denoted by $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{W}^{s,p}(\Omega) := (W^{s,p}(\Omega))^d$ and $\mathbf{H}^s(\Omega) := (H^s(\Omega))^d$. Let us point that by Theorem 1.2 of [32] for any $\mathbf{u} \in \mathbf{L}^2(\Omega)$, satisfying $\operatorname{div}(\mathbf{u}) = 0$ in $\mathcal{D}'(\Omega)$, the normal component of \mathbf{u} , i.e. $\mathbf{u}_{\mathbf{n}} := \mathbf{u} \cdot \mathbf{n}$, exists and belongs to $H^{-1/2}(\Gamma)$. We will also use the following divergence free spaces

$$\begin{aligned} \mathbf{V}^s(\Omega) &:= \{\mathbf{u} \in \mathbf{H}^s(\Omega) : \operatorname{div}(\mathbf{u}) = 0 \text{ in } \mathcal{D}'(\Omega), \int_{\Gamma} \mathbf{u}_{\mathbf{n}} \, d\mathbf{x} = 0\}, \\ \mathbf{V}^s(\Gamma) &:= \{\mathbf{u} \in \mathbf{H}^s(\Gamma) : \int_{\Gamma} \mathbf{u}_{\mathbf{n}} \, d\mathbf{x} = 0\}, \quad \mathbf{V}^{-s}(\Gamma) := (\mathbf{V}^s(\Gamma))' \end{aligned}$$

and

$$\mathbf{G}(\Gamma_T) := \left\{ \mathbf{u} \in L^2(0, T; \mathbf{V}^{1/2}(\Gamma)) : \partial_t \mathbf{u} \in L^2(0, T; \mathbf{V}^{-1/2}(\Gamma)) \right\}.$$

Let us formulate some results for the Stokes equations, used in the sequel. We consider the system

$$\begin{cases} -\nu \Delta \mathbf{v} = -\nabla p, & \operatorname{div}(\mathbf{v}) = 0 \text{ in } \Omega, \\ \mathbf{v} = \mathbf{b} \text{ on } \Gamma. \end{cases} \quad (3.11)$$

The proof of the following result has been done by Cattabriga in [3] (see also Theorem 3 with Remarks 2 of [11]).

Proposition 3.1. *If $\mathbf{b} \in \mathbf{H}^{s-1/2}(\Gamma)$ for $s = 0$ or $s = 1$, then there exists an unique weak solution $\mathbf{v} \in \mathbf{V}^s(\Omega)$ of (3.11), such that*

$$\|\mathbf{v}\|_{\mathbf{V}^s(\Omega)} \leq C \|\mathbf{b}\|_{\mathbf{H}^{s-1/2}(\Gamma)}.$$

4 Statement of the Stokes B - L system

In this section we are going to formulate the mathematical problem. Let us assume that τ, ν are given positive fixed parameters. We are concerned with the following initial-boundary value problem, denoted as \mathbf{IBVP}_{τ} :

Find a pair $(u, \mathbf{v}) = (u(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x})) : \Omega_T \rightarrow \mathbb{R} \times \mathbb{R}^d$ solution to the Stokes-Buckley-Leverett system in the domain Ω_T

$$\partial_t u + \operatorname{div}(\mathbf{v} g(u)) = 0, \quad (4.12)$$

$$\tau \partial_t \mathbf{v} - \nu \Delta \mathbf{v} + h(u) \mathbf{v} = -\nabla p, \quad \operatorname{div}(\mathbf{v}) = 0, \quad (4.13)$$

satisfying the boundary conditions

$$(u, \mathbf{v}) = (u_b, \mathbf{b}) \quad \text{on } \Gamma_T, \quad (4.14)$$

and the initial conditions

$$(u, \mathbf{v})|_{t=0} = (u_0, \mathbf{v}_0) \quad \text{in } \Omega. \quad (4.15)$$

We assume that our data satisfy the following regularity properties

$$\begin{aligned} g, h &\in W_{\text{loc}}^{1,\infty}(\mathbb{R}) \quad \text{with } 0 < h_0 \leq h(u), \\ 0 &\leq u_b \leq 1 \quad \text{on } \Gamma_T, \\ 0 &\leq u_0 \leq 1 \quad \text{on } \Omega, \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} \mathbf{v}_0 &\in \mathbf{V}^0(\Omega) \quad \text{and} \quad \mathbf{b} \in \mathbf{G}(\Gamma_T), \quad \text{such that} \\ \mathbf{b}(0) \cdot \mathbf{n} &= \mathbf{v}_0 \cdot \mathbf{n} \quad \text{in } H^{-1/2}(\Gamma). \end{aligned} \quad (4.17)$$

Now, since equation (4.12) is a hyperbolic scalar conservation law, the saturation function u may admit shocks. Therefore, in order to select the more correct physical solution, we need the entropy concept as given at the following

Definition 4.1. A pair $\mathbf{F}(u) := (\eta(u), q(u))$ is called an entropy pair for (4.12), if $\eta : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous and also convex function and the function $q : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$q'(u) = \eta'(u) g'(u) \quad \text{for a. a. } u \in \mathbb{R}. \quad (4.18)$$

Certainly, the most important example of entropy pairs are given by the Kruřkov's entropies. Here, we consider the following parameterized family of Kruřkov's entropy pairs for (4.12)

$$\mathbf{F}(u, v) = \left(|u - v|, \operatorname{sgn}(u - v)(g(u) - g(v)) \right) \quad (4.19)$$

for each $v \in \mathbb{R}$. We remark that any smooth entropy pair $\mathbf{F}(u) := (\eta(u), q(u))$ for (4.12) can be recovered by the family given by (4.19). The inverse one is also true, i.e. any entropy pair $\mathbf{F}(u) := (\eta(u), q(u))$ given by (4.19) can be recovered by a family of smooth entropy pairs. In fact, this result follows for any entropy by a standard regularization argument.

Another two examples of parameterized family of entropy pairs for (4.12) are

$$\mathbf{F}^\pm(u, v) = \left(|u - v|^\pm, \operatorname{sgn}^\pm(u - v)(g(u) - g(v)) \right) \quad (4.20)$$

for each $v \in \mathbb{R}$, which will be useful in the Kinetic formulation (see Section 5.2). Here

$$\operatorname{sgn}^+(v) := \begin{cases} 1, & \text{if } v > 0, \\ 0, & \text{if } v \leq 0, \end{cases} \quad \operatorname{sgn}^-(v) := \begin{cases} 0, & \text{if } v < 0, \\ 1, & \text{if } v \geq 0 \end{cases}$$

and $|v|^\pm := \max\{\pm v, 0\}$, respectively.

The following definition tells us in which sense a pair of functions (u, \mathbf{v}) is a weak solution of **IBVP** $_\tau$: (4.12)-(4.15).

Definition 4.2. *A pair of functions*

$$(u, \mathbf{v}) \in L^\infty(\Omega_T) \times L^2(0, T; \mathbf{V}^1(\Omega))$$

*is called a weak solution to the **IBVP** $_\tau$, if the pair (u, \mathbf{v}) satisfies the integral inequality*

$$\begin{aligned} & \iint_{\Omega_T} (|u - v| \phi_t + \operatorname{sgn}(u - v)(g(u) - g(v)) \mathbf{v} \cdot \nabla \phi) \, d\mathbf{x} \, dt \\ & + \int_{\Gamma_T} M |u_b - v| \phi \, d\mathbf{x} \, dt + \int_{\Omega} |u_0 - v| \phi(0, x) \, d\mathbf{x} \geq 0 \end{aligned} \quad (4.21)$$

for any fixed $v \in \mathbb{R}$, where $M := K|\mathbf{b}_n|$ defined on Γ_T with $K := \|g'\|_{L^\infty(\mathbb{R})}$ and for any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^d)$ and also the following integral identity

$$\int_{\Omega_T} [\tau \mathbf{v} \cdot \psi_t - \nu \nabla \mathbf{v} : \nabla \psi - h(u) \mathbf{v} \cdot \psi] \, d\mathbf{x} \, dt + \tau \int_{\Omega} \mathbf{v}_0 \cdot \psi(0) \, d\mathbf{x} = 0 \quad (4.22)$$

holds for any $\psi \in \mathbf{C}^1(\overline{\Omega_T})$, such that $\psi = 0$ at $t = T$ and on Γ_T . Moreover the trace of \mathbf{v} is equal to \mathbf{b} on Ω . Here $\nabla \mathbf{v} : \nabla \psi = \sum_{j=1}^d \partial_{x_j} \mathbf{v} \cdot \partial_{x_j} \psi$.

For more complete discussions on this concept of weak entropy solutions for hyperbolic conservation law (4.12) (with boundary conditions), we refer to Otto [22], Neves [21], Chen, Frid [4] (see Theorem 4.1) and Malek et al [20] (see Lemma 7.24 and Theorem 7.31), further the Dafermos' treatise book [7].

Theorem 4.3. *If the data $g, h, u_b, u_0, \mathbf{v}_0, \mathbf{b}$ fulfills the regularity properties (4.16)-(4.17), then the **IBVP** $_\tau$ has a weak solution*

$$(u, \mathbf{v}) \in L^\infty(\Omega_T) \times L^2(0, T; \mathbf{V}^1(\Omega)) \cap H^1(0, T; \mathbf{V}^{-1}(\Omega)),$$

satisfying

$$0 \leq u \leq 1 \quad \text{a . e. in } \Omega_T,$$

$$\|\sqrt{\tau} \mathbf{v}\|_{C([0, T]; \mathbf{V}^0(\Omega))} + \|\mathbf{v}\|_{L^2(0, T; \mathbf{V}^1(\Omega))} + \tau \|\mathbf{v}\|_{H^1(0, T; \mathbf{V}^{-1}(\Omega))} \leq C, \quad (4.23)$$

where C is a positive constant independent of τ .

5 Existence of weak solution

5.1 Parabolic approximation

In order to show the existence of a weak solution for the **IBVP** $_{\tau}$, first we study the following approximated parabolic system with a fixed parameter $\varepsilon > 0$

$$\partial_t u^\varepsilon + \operatorname{div}(\mathbf{v}^\varepsilon g(u^\varepsilon)) = \varepsilon \Delta u^\varepsilon \quad \text{in } \Omega_T, \quad (5.24)$$

$$\tau \partial_t \mathbf{v}^\varepsilon - \nu \Delta \mathbf{v}^\varepsilon + h(u^\varepsilon) \mathbf{v}^\varepsilon = -\nabla p^\varepsilon, \quad \operatorname{div}(\mathbf{v}^\varepsilon) = 0 \quad \text{in } \Omega_T \quad (5.25)$$

jointly with the boundary-initial conditions

$$\begin{aligned} \varepsilon \frac{\partial u^\varepsilon}{\partial \mathbf{n}} + M(u^\varepsilon - u_b^\varepsilon) &= 0 \quad \text{and} \quad \mathbf{v}^\varepsilon = \mathbf{b} \quad \text{on } \Gamma_T \\ (u^\varepsilon, \mathbf{v}^\varepsilon)|_{t=0} &= (u_0^\varepsilon, \mathbf{v}_0) \quad \text{in } \Omega, \end{aligned} \quad (5.26)$$

where $u_b^\varepsilon, u_0^\varepsilon$ are regularized boundary-initial data satisfying suitable compatibility conditions. We remark that $u_b^\varepsilon, u_0^\varepsilon$ converge strongly in $L^1_{\text{loc}}(\Gamma_T)$ and $L^1_{\text{loc}}(\Omega)$, respectively, to u_b, u_0 .

In the section 5.2.2 we establish the following result.

Proposition 5.1. *For each $\varepsilon > 0$, there exists a unique solution $(u^\varepsilon, \mathbf{v}^\varepsilon)$ of the system (5.24)–(5.26), which has the following regularity $u^\varepsilon \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ and $\mathbf{v}^\varepsilon \in L^2(0, T; \mathbf{V}^1(\Omega)) \cap H^1(0, T; \mathbf{V}^{-1}(\Omega))$ satisfying*

$$\varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega_T)}^2 \leq C \quad \text{and} \quad 0 \leq u^\varepsilon \leq 1 \quad \text{a.e. on } \Omega_T, \quad (5.27)$$

$$\|\sqrt{\tau} \mathbf{v}^\varepsilon\|_{C([0, T]; \mathbf{V}^0(\Omega))} + \|\mathbf{v}^\varepsilon\|_{L^2(0, T; \mathbf{V}^1(\Omega))} + \|\tau \mathbf{v}^\varepsilon\|_{H^1(0, T; \mathbf{V}^{-1}(\Omega))} \leq C, \quad (5.28)$$

where C is a positive constant independent of ε (and τ).

Remark 5.2. *After obtaining that $0 \leq u^\varepsilon \leq 1$ we can consider that $g(s) := g(0)$, $h(s) := h(0)$ for any $s < 0$ and $g(s) := g(1)$, $h(s) := h(1)$ for any $s > 1$.*

5.2 The limit transition on $\varepsilon \rightarrow 0$

In this section we are concerned to pass to the limit in (5.24)–(5.25) as $\varepsilon \rightarrow 0$. Since this problem is non-linear on u^ε , the estimates (5.27)–(5.28) are not sufficient to take the limit transition on ε as it goes to 0. In fact, we need a strong convergence of a subsequence for the family $\{u^\varepsilon\}_{\varepsilon > 0}$. Then, to derive the strong convergence for u^ε , we use the Theory of Kinetic Formulation as introduced by Lions, Perthame and Tadmor [17]–[18], [26]. Here, we are going to follow closer Perthame and Dalibard [25]. That is, first we take the Kinetic formulation of (5.24)–(5.25), then we pass to the weak limit. Finally, the information that the initial-boundary conditions converge strongly, we are able to show the strong convergence of u^ε .

5.2.1 The main idea of the limit transition. Sketch of the proof

Let $(\eta(u), q(u))$ be an entropy pair for (4.12). Then, we have in distribution sense

$$\partial_t \eta(u^\varepsilon) + \operatorname{div}(\mathbf{v}^\varepsilon q(u^\varepsilon)) - \varepsilon \Delta \eta(u^\varepsilon) = -\varepsilon \eta''(u) |\nabla \eta(u)|^2 \leq 0,$$

since η is a convex function. For instance, we could take the entropy pair $(\eta(u), q(u)) = \mathbf{F}^+(u, v)$ for all $v \in \mathbb{R}$, defined by (4.20). Then, we have in sense of distributions

$$\partial_t |u^\varepsilon - v|^+ + \operatorname{div}[\mathbf{v}^\varepsilon \operatorname{sgn}^+(u^\varepsilon - v)(g(u^\varepsilon) - g(v))] - \varepsilon \Delta |u^\varepsilon - v|^+ = -m^\varepsilon, \quad (5.29)$$

where m^ε is a real nonnegative Radon measure.

If we differentiate in the distribution sense (5.29) with respect to v , we get (as now a standard procedure in the kinetic theory) the following transport equation

$$\partial_t f^\varepsilon + g'(v) \mathbf{v}^\varepsilon \cdot \nabla f^\varepsilon - \varepsilon \Delta f^\varepsilon = \partial_v m^\varepsilon, \quad (5.30)$$

where $f^\varepsilon(t, \mathbf{x}, v) := \operatorname{sgn}^+(u^\varepsilon(t, \mathbf{x}) - v)$. Let us point out that

$$0 \leq f^\varepsilon(t, \mathbf{x}, v) \leq 1 \quad \text{in } \Omega_T \times \mathbb{R}.$$

Later on we show that m^ε is uniformly bounded with respect to ε , hence using (5.27)-(5.28) there exist subsequences of the families $\{m^\varepsilon, f^\varepsilon, \mathbf{v}^\varepsilon\}$ and a real nonnegative Radon measure $m = m(t, \mathbf{x}, v)$, functions $f \in L^\infty(\Omega_T \times \mathbb{R})$ and $\mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$, such that

$$\begin{aligned} m^\varepsilon &\rightarrow m && \text{weakly in } \mathcal{M}(\Omega_T \times \mathbb{R}), \\ f^\varepsilon &\rightarrow f && \star\text{-weakly in } L^\infty(\Omega_T \times \mathbb{R}), \\ \mathbf{v}^\varepsilon &\rightarrow \mathbf{v} && \text{strongly in } L^2(\Omega_T). \end{aligned}$$

Since (5.30) is linear, it follows that

$$\partial_t f + g'(v) \mathbf{v} \cdot \nabla f = \partial_v m \quad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \quad (5.31)$$

Accounting the initial boundary conditions for f^ε , we also obtain

$$f = \operatorname{sgn}^+(u_0 - v) \quad \text{for } t = 0 \quad \text{and} \quad f = \operatorname{sgn}^+(u_b - v) \quad (5.32)$$

on the influx part of $\Gamma_T \times \mathbb{R}$, i.e. where $g'(v) \mathbf{b}_n < 0$. By the regularity of the velocity field $\mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$, we can use the theory for transport equations, introduced by DiPerna-Lions [9], and deduce that the solution of (5.31)-(5.32) takes values equals only to 0 and 1 on $\Omega_T \times \mathbb{R}$. Since $f(\cdot, \cdot, v)$ is a monotone function on v (as a limit of $f^\varepsilon(\cdot, \cdot, v)$ being monotone one too), we have

$$f = \operatorname{sgn}^+(z(t, \mathbf{x}) - v) \quad \text{for some } z = z(t, \mathbf{x}).$$

Finally, simplest considerations will apply that $z(t, \mathbf{x}) \equiv u(t, \mathbf{x})$ and we have a strong convergence of u^ε to u , that ends the proof of our convergence result.

5.2.2 Proof of Proposition 5.1

Let $(\eta(u), q(u))$ be an entropy pair, satisfying the condition

$$|q(u)| \leq K\eta(u) \quad \text{for } u \in \mathbb{R}. \quad (5.33)$$

Both the pairs $\mathbf{F}^\pm(u, v)$ for any $v \in \mathbb{R}$, defined by (4.20), as the pair $\eta(u) = u^2$, $q(u) = \int_0^u 2s g'(s) ds$ fulfill this condition.

If we multiply (5.24) by $\eta'(u^\varepsilon)\phi$ with a function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^d)$ and integrate on Ω_T , we obtain

$$\begin{aligned} & \iint_{\Omega_T} [\eta(u^\varepsilon)\phi_t + q(u^\varepsilon)(\mathbf{v}^\varepsilon \cdot \nabla)\phi - \varepsilon(\nabla\phi \cdot \nabla\eta(u^\varepsilon))] d\mathbf{x}dt \\ & + \int_{\Omega} \eta(u_0^\varepsilon)\phi(0) d\mathbf{x} + \int_{\Gamma_T} M\eta(u_b^\varepsilon)\phi d\mathbf{x}dt = m_\eta^\varepsilon(\phi), \end{aligned} \quad (5.34)$$

where

$$\begin{aligned} m_\eta^\varepsilon(\phi) &:= \iint_{\Omega_T} \varepsilon\eta''|\nabla u^\varepsilon|^2\phi dtd\mathbf{x} + \int_{\Gamma_T} \{\mathbf{b}_n q(u^\varepsilon) + M\eta(u^\varepsilon) \\ &+ \frac{1}{2}M\eta''(r)(u_b^\varepsilon - u^\varepsilon)^2\}\phi dtd\mathbf{x}. \end{aligned} \quad (5.35)$$

Here we used that $\eta(u_b^\varepsilon) = \eta(u^\varepsilon) + \eta'(u^\varepsilon)(u_b^\varepsilon - u^\varepsilon) + \frac{\eta''(r)}{2}(u_b^\varepsilon - u^\varepsilon)^2$ for some function r with values between u^ε and u_b^ε a.e. on Γ_T . Let us observe that

$$m_\eta^\varepsilon(\phi) \geq \int_{\Omega_T} \varepsilon\eta''|\nabla u^\varepsilon|^2\phi dtd\mathbf{x} \geq 0 \quad \text{for } \phi \geq 0. \quad (5.36)$$

Choosing in (5.34) $\phi(t, \mathbf{x}) := 1 - \zeta_\delta(t - t_0)$ for $t_0 \in (0, T)$ with

$$\zeta_\delta(s) := \begin{cases} 0, & \text{if } s < 0 \text{ and } 1, & \text{if } s > \delta, \\ \frac{s}{\delta}, & \text{if } 0 \leq s \leq \delta \end{cases} \quad (5.37)$$

and passing to the limit on $\delta \rightarrow 0^+$, we derive

$$\int_{\Omega} \eta(u^\varepsilon) d\mathbf{x}(t_0) + \int_0^{t_0} \int_{\Omega} \varepsilon\eta''|\nabla u^\varepsilon|^2 dtd\mathbf{x} \leq \int_{\Omega} \eta(u_0^\varepsilon) d\mathbf{x} + \int_0^{t_0} \int_{\Gamma} M\eta(u_b^\varepsilon) dtd\mathbf{x}.$$

Hence taking $\eta(u) = |u|^-$ ($\eta = |u - 1|^+$ and $\eta = u^2$, consistently) in this inequality, we obtain the estimates (5.27) by the regularity assumptions (4.16). The regularity $u^\varepsilon \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ follows from the well-known theory for parabolic type equations (see Ladyzhenskaya et al [15]).

Now let us consider the quasi-stationary Stokes type equations

$$\begin{cases} -\nu\Delta \mathbf{v}_b = -\nabla p_b, & \text{div}(\mathbf{v}_b) = 0 \quad \text{in } \Omega_T, \\ \mathbf{v}_b = \mathbf{b} \quad \text{on } \Gamma_T. \end{cases} \quad (5.38)$$

In view of Proposition 3.1 and the assumption (4.17), the solution \mathbf{v}_b of this problem exists and fulfills the estimate

$$\|\mathbf{v}_b\|_{L^2(0,T;\mathbf{H}^1(\Omega))} + \|\partial_t \mathbf{v}_b\|_{\mathbf{L}^2(\Omega_T)} \leq C. \quad (5.39)$$

Therefore taking the difference between (5.25) and (5.38), we have that the function $\mathbf{w}^\varepsilon = \mathbf{v}^\varepsilon - \mathbf{v}_b$ satisfies

$$\begin{aligned} \tau \partial_t \mathbf{w}^\varepsilon - \nu \Delta \mathbf{w}^\varepsilon + h(u^\varepsilon) \mathbf{w}^\varepsilon &= -\nabla(P^\varepsilon) + \mathbf{f}^\varepsilon, & \operatorname{div}(\mathbf{w}^\varepsilon) &= 0, \\ \mathbf{w}^\varepsilon|_{\Gamma_T} &= 0, & \mathbf{w}^\varepsilon|_{t=0} &= \mathbf{v}_0 - \mathbf{v}_b|_{t=0}, \end{aligned}$$

with $P^\varepsilon := p^\varepsilon - p_b$, $\mathbf{f}^\varepsilon := -\tau \partial_t \mathbf{v}_b - h(u^\varepsilon) \mathbf{v}_b$. Let us point that the solvability of the above system can be shown as in [14], [32]. If we multiply the first equation in this system by \mathbf{w}^ε and integrate over Ω , we obtain

$$\begin{aligned} \frac{d}{dt} \left(\frac{\tau}{2} \|\mathbf{w}^\varepsilon\|_{\mathbf{L}^2(\Omega)}^2 \right) + \nu \|\nabla \mathbf{w}^\varepsilon\|_{\mathbf{L}^2(\Omega)}^2 &\leq \int_{\Omega} |(\mathbf{f}^\varepsilon \cdot \mathbf{w}^\varepsilon)| \, d\mathbf{x} \\ &\leq \frac{\nu}{2} \|\nabla \mathbf{w}^\varepsilon\|_{\mathbf{L}^2(\Omega)}^2 + C \|\mathbf{f}^\varepsilon\|_{\mathbf{L}^2(\Omega)}^2, \end{aligned}$$

where we have used Poincaré's inequality.

Then, using (4.17), (5.39) we deduce

$$\tau \|\mathbf{w}^\varepsilon\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}^2 + \|\mathbf{w}^\varepsilon\|_{L^2(0,T;\mathbf{V}^1(\Omega))}^2 \leq C$$

with some constant C independent of ε (and τ). Hence in view of the weak formulation (4.22) of (5.25) and Lemmas 1.2-1.4, p.176 of [32], we get that $\mathbf{w}^\varepsilon \in C([0, T]; \mathbf{V}^0(\Omega))$ and the estimate (5.28).

Finally, with the help of derived estimates (5.27)-(5.28), we can apply Leray-Schauder's fixed point argument (as now a standard procedure) and get the solvability of the approximated system (5.24)-(5.26). \blacksquare

5.2.3 Rigorous proof of the limit transition

Now, if we take in (5.34) the entropy pair $\mathbf{F}^+(u, v)$ for all $v \in \mathbb{R}$, then we see that the function $f^\varepsilon(t, \mathbf{x}, v) = \operatorname{sgn}^+(u^\varepsilon - v)$ satisfies for all nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^d)$, the following identity

$$\begin{aligned} \iint_{\Omega_T} \left\{ \int_v^1 f^\varepsilon(t, \mathbf{x}, s) [\phi_t + g'(s) (\mathbf{v}^\varepsilon \cdot \nabla) \phi] \, ds - \varepsilon \nabla \phi \cdot \nabla |u^\varepsilon - v|^+ \right\} d\mathbf{x} dt \\ + \int_{\Omega} |u_0^\varepsilon - v|^+ \phi(0) d\mathbf{x} + \int_{\Gamma_T} M |u_b^\varepsilon - v|^+ \phi d\mathbf{x} dt = m_+^\varepsilon(\phi) \geq 0, \quad (5.40) \end{aligned}$$

where $m_+^\varepsilon := m_{|u^\varepsilon - v|^+}^\varepsilon$ (see (5.35) and (5.36)).

Further, we have for any $G \in C^1([0, 1])$, with $G(0) = 0$ that

$$\begin{aligned} G(u^\varepsilon) &= \int_0^1 G'(s) f^\varepsilon(\cdot, \cdot, s) ds \quad \text{a. e. in } \Omega_T, \\ 0 &\leq f^\varepsilon \leq 1 \quad \text{on } \Omega_T \times \mathbb{R}, \quad f^\varepsilon(t, \mathbf{x}, v) = \begin{cases} 1, & \text{for } v \leq 0, \\ 0, & \text{for } v \geq 1, \end{cases} \\ \partial_v f^\varepsilon &\leq 0 \quad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \end{aligned} \quad (5.41)$$

Let us choose in (5.40) $\phi := 1 - \zeta_\delta(t - T + \delta)$ and then, passing to the limit as $\delta \rightarrow 0^+$, we get

$$m_+^\varepsilon(1) \leq \int_\Omega |u_0^\varepsilon - v|^+ d\mathbf{x} + \int_{\Gamma_T} M |u_b^\varepsilon - v|^+ d\mathbf{x} dt, \quad (5.42)$$

hence the Riesz representation theorem implies that the real positive Radon measure m_+^ε is well defined on $\overline{\Omega}_T \times \mathbb{R}$, such that

$$\begin{aligned} m_+^\varepsilon(\cdot, \cdot, v) &= 0 \quad \text{for any } v > 1 \quad \text{on } \Omega_T, \text{ and} \\ \iint_{\overline{\Omega}_T} m_+^\varepsilon(t, \mathbf{x}, v) d\mathbf{x} dt &\leq C(v) \quad \text{for all finite } v, \end{aligned} \quad (5.43)$$

where $C(v)$ is a positive constant independent of ε , but could depend on v . By a similar way as (5.40) has been derived, if we take the entropy pair $\mathbf{F}^-(u, v)$ for all $v \in \mathbb{R}$, defined by (4.20), we obtain

$$\begin{aligned} &\iint_{\Omega_T} \left\{ \int_0^v (1 - f^\varepsilon(t, \mathbf{x}, s)) [\phi_t + g'(s) (\mathbf{v}^\varepsilon \cdot \nabla) \phi] dv - \varepsilon \nabla \phi \cdot \nabla |u^\varepsilon - v|^- \right\} d\mathbf{x} dt \\ &+ \int_\Omega |u_0^\varepsilon - v|^- \phi(0) d\mathbf{x} + \int_{\Gamma_T} M |u_b^\varepsilon - v|^- \phi d\mathbf{x} dt = m_-^\varepsilon(\phi) \geq 0, \end{aligned} \quad (5.44)$$

where $m_-^\varepsilon := m_{|u^\varepsilon - v|^-}^\varepsilon$. Moreover we have that the real positive Radon measure m_-^ε , defined on $\overline{\Omega}_T \times \mathbb{R}$, fulfills the following properties

$$\begin{aligned} m_-^\varepsilon(1) &\leq \int_\Omega |u_0^\varepsilon - v|^- d\mathbf{x} + \int_{\Gamma_T} M |u_b^\varepsilon - v|^- d\mathbf{x} dt, \\ m_-^\varepsilon(\cdot, \cdot, v) &= 0 \quad \text{for any } v < 0 \text{ and on } \overline{\Omega}_T, \\ \iint_{\overline{\Omega}_T} m_-^\varepsilon(t, \mathbf{x}, v) d\mathbf{x} dt &\leq C(v) \quad \text{for all finite } v. \end{aligned} \quad (5.45)$$

In view of Proposition 5.1 and (5.43), (5.45), there exist subsequences of f^ε , \mathbf{v}^ε , m^ε and the functions

$$f \in L^\infty(\Omega_T \times \mathbb{R}), \quad \mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega)), \quad (5.46)$$

and a real nonnegative Radon measure $m = m(t, \mathbf{x}, v)$, such that

$$\begin{aligned} f^\varepsilon &\rightarrow f \quad \star\text{-weakly in } L^\infty(\Omega_T \times \mathbb{R}), \\ \mathbf{v}^\varepsilon &\rightarrow \mathbf{v}, \quad \varepsilon \nabla u^\varepsilon \rightarrow 0 \quad \text{strongly in } L^2(\Omega_T), \\ m_\pm^\varepsilon &\rightarrow m_\pm \quad \text{weakly in } \mathcal{M}_{loc}^+(\overline{\Omega}_T \times \mathbb{R}). \end{aligned}$$

Now, for any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^{d+1})$ the following integral inequalities fulfills

$$\begin{aligned} & \iint_{\Omega_T} \int_v^1 f(t, \mathbf{x}, s) [\phi_t + g'(s) (\mathbf{v} \cdot \nabla) \phi] ds d\mathbf{x} dt + \int_{\Omega} |u_0 - v|^+ \phi(0) d\mathbf{x} \\ & + \int_{\Gamma_T} M |u_b - v|^+ \phi d\mathbf{x} dt = m_+(\phi) =: \int_{\Omega_T} m_+(t, \mathbf{x}, v) \phi d\mathbf{x} dt \geq 0, \end{aligned} \quad (5.47)$$

and

$$\begin{aligned} & \iint_{\Omega_T} \int_0^v (1 - f(t, \mathbf{x}, s)) [\phi_t + g'(s) (\mathbf{v} \cdot \nabla) \phi] ds d\mathbf{x} dt + \int_{\Omega} |u_0 - v|^- \phi(0) d\mathbf{x} \\ & + \int_{\Gamma_T} M |u_b - v|^- \phi d\mathbf{x} dt = m_-(\phi) =: \int_{\Omega_T} m_-(t, \mathbf{x}, v) \phi d\mathbf{x} dt \geq 0. \end{aligned} \quad (5.48)$$

Moreover, we have for any $G \in C^1([0, 1])$, with $G(0) = 0$ that

$$\begin{aligned} G(u) &= \int_0^1 G'(s) f(\cdot, \cdot, s) ds \quad \text{a.e. in } \Omega_T, \\ 0 &\leq f \leq 1 \quad \text{on } \Omega_T \times \mathbb{R}, \quad f(t, \mathbf{x}, v) = \begin{cases} 1, & \text{for } v \leq 0, \\ 0, & \text{for } v \geq 1, \end{cases} \\ \partial_v f &\leq 0 \quad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}), \end{aligned} \quad (5.49)$$

and

$$\begin{aligned} \iint_{\Omega_T} m_{\pm}(t, \mathbf{x}, v) d\mathbf{x} dt &\leq C(v) \quad \text{for all finite } v, \\ m_+(\cdot, \cdot, v) &= 0 \quad \text{for any } v > 1 \text{ and on } \overline{\Omega}_T, \\ m_-(\cdot, \cdot, v) &= 0 \quad \text{for any } v < 0 \text{ and on } \overline{\Omega}_T, \\ m_{\pm}(\cdot, \cdot, v) &\in C(\mathbb{R}; \mathcal{M}^+(\overline{\Omega}_T \times \mathbb{R})), \end{aligned} \quad (5.50)$$

the continuity of $m_{\pm}(\cdot, \cdot, v)$ on v follows from the left parts of (5.47), (5.48).

Finally, taking in (5.47) and (5.48) $\phi = \partial_v \psi$, with ψ being a nonnegative function in $C_0^\infty(\Omega_T \times \mathbb{R})$, integrating by parts on v , we obtain that f satisfies the following transport equations

$$\begin{aligned} \iint_{\Omega_T \times \mathbb{R}} f [\psi_t + g'(v) (\mathbf{v} \cdot \nabla) \psi] - m_+ \partial_v \psi dv d\mathbf{x} dt &= 0, \\ \iint_{\Omega_T \times \mathbb{R}} (1 - f) [\psi_t + g'(v) (\mathbf{v} \cdot \nabla) \psi] + m_- \partial_v \psi dv d\mathbf{x} dt &= 0, \end{aligned} \quad (5.51)$$

respectively.

Now, let us study the trace concept on the initial-boundary terms.

Proposition 5.3. *The function $f = f(t, \mathbf{x}, v)$ has the trace f^0 at the time $t = 0$, such that*

$$f^0 = f^0(\mathbf{x}, v) \equiv \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta f(s, \mathbf{x}, v) ds$$

and

$$f^0 = (f^0)^2 \quad \text{a. a. on } \Omega \times \mathbb{R}. \quad (5.52)$$

The function $f = f(t, \mathbf{x}, v)$ has the trace f^b on $\Gamma_T \times \mathbb{R}$, such that

$$f^b = f^b(t, \mathbf{x}, v) \equiv \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta f(t, \mathbf{x} - s \mathbf{n}(\mathbf{x}), v) ds$$

for a. a. $(t, \mathbf{x}, v) \in \Gamma_T \times \mathbb{R}$, where $g'(v) \mathbf{b}_n(t, \mathbf{x}) \neq 0$ and

$$f^b = (f^b)^2 \quad (5.53)$$

for a. a. $(t, \mathbf{x}, v) \in \Gamma_T \times \mathbb{R}$, where $g'(v) \mathbf{b}_n(t, \mathbf{x}) < 0$.

Proof. First, let $\gamma \in C_0^\infty(\mathbb{R})$ be a fixed function. Then by (5.51) the vector function

$$\mathbf{f}_\gamma := \left(\int_R f(\cdot, \cdot, v) \gamma(v) dv, \quad \mathbf{v} \int_R g'(v) f(\cdot, \cdot, v) \gamma(v) dv \right) \in \mathbf{L}^2(\Omega_T),$$

and it follows that

$$\operatorname{div}_{(t, \mathbf{x})}(\mathbf{f}_\gamma) = - \int_R m_+(\cdot, \cdot, v) \gamma'(v) dv \in \mathcal{M}(\Omega_T)$$

having a finite total variation $|\operatorname{div}_{(t, \mathbf{x})} \mathbf{f}_\gamma|(\Omega_T) < \infty$, in view of (5.50). Let Σ and $\mathbf{n}_{(t, \mathbf{x})}$ be the boundary of $\overline{\Omega}_T$ and the unitary normal to Σ , respectively. By Theorem 2.1 in Chen & Frid [4], $\mathbf{f}_\gamma \cdot \mathbf{n}_{(t, \mathbf{x})}$ is a continuous linear functional over $H^{1/2}(\Sigma) \cap L^\infty(\Sigma)$.

1.1 Now, if we take $\varphi \in C_0^\infty((-\infty, T) \times \Omega)$, then

$$\begin{aligned} \langle \mathbf{f}_\gamma \cdot \mathbf{n}_{(t, \mathbf{x})} |_{t=0}, \varphi \rangle &= \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta \int_\Omega \left[\int_{\mathbb{R}} f(t, \mathbf{x}, v) \gamma(v) dv \right] \varphi(t, \mathbf{x}) d\mathbf{x} dt \\ &= \lim_{\delta \rightarrow 0^+} \iint_{\Omega \times \mathbb{R}} \left[\frac{1}{\delta} \int_0^\delta f(t, \mathbf{x}, v) dt \right] \gamma(v) \varphi(0, \mathbf{x}) d\mathbf{x} dv. \end{aligned}$$

Since $0 \leq \frac{1}{\delta} \int_0^\delta f(t, \cdot, \cdot) dt \leq 1$ on $\Omega \times \mathbb{R}$, using the dominated convergence theorem we derive the existence of $\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta f(s, \cdot, \cdot) ds$, which we denote by f^0 . It is obvious that $0 \leq f^0 \leq 1$ on $\Omega \times \mathbb{R}$ and $\mathbf{f}_\gamma \cdot \mathbf{n}_{(t, \mathbf{x})} |_{t=0} = \langle \mathbf{f}^0, \gamma \rangle$ a.e. on Ω . Since γ is an arbitrary function, therefore we can simply denote $\mathbf{f} \cdot \mathbf{n}_{(t, \mathbf{x})} |_{t=0} = f^0$.

1.2. Let us take a nonnegative function $\psi \in C_0^\infty(\Omega)$ and set $\varphi(t, \mathbf{x}) = \zeta_\delta(t)\psi(\mathbf{x})$ in inequalities (5.47), (5.48) with ζ_δ given by (5.37). Then, we obtain, after passing to the limit as $\delta \rightarrow 0^+$, respectively

$$\int_{\Omega} \psi(\mathbf{x}) \left[- \int_v^1 f^0(\mathbf{x}, s) ds d\mathbf{x} + |u_0(\mathbf{x}) - v|^+ \right] d\mathbf{x} \geq 0,$$

and

$$\int_{\Omega} \psi(\mathbf{x}) \left[- \int_0^v (1 - f^0(\mathbf{x}, s)) ds d\mathbf{x} + |u_0(\mathbf{x}) - v|^- \right] d\mathbf{x} \geq 0.$$

Since ψ is an arbitrary nonnegative function, then for a. a. $\mathbf{x} \in \Omega$ the 1st inequality implies that $f^0(\mathbf{x}, v) = 0$ if $v > u_0(\mathbf{x})$ and the 2nd one gives $f^0(\mathbf{x}, v) = 1$ if $v < u_0(\mathbf{x})$, i.e. we show (5.52).

2.1. Let $d(\mathbf{x}) := \min_{\mathbf{y} \in \Gamma} |\mathbf{x} - \mathbf{y}|$ be the distance function from $\mathbf{x} \in \overline{\Omega}$ to Γ . Denoting by $\mathbf{x}_s := \mathbf{x} - s\mathbf{n}(\mathbf{x})$, for any $\mathbf{x} \in \Gamma$ and $s \in (0, \delta)$, and applying again the result of Theorem 2.1 in [4], we have that for any $\psi \in C_0^\infty((0, T) \times \mathbb{R}^d)$

$$\begin{aligned} & \langle \mathbf{f}_\gamma \cdot \mathbf{n}|_{\Gamma_T}, \psi \rangle \\ &= \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta \int_{\Gamma_T} \left[\int_{\mathbb{R}} g'(v) (\mathbf{v} \cdot (-\nabla d))(t, \mathbf{x}_s) f(t, \mathbf{x}_s, v) \gamma(v) dv \right] \psi(t, \mathbf{x}_s) dx dt ds \\ &= \int_{\Gamma_T \times \mathbb{R}} g'(v) \mathbf{b}_n(t, \mathbf{x}) \psi(t, \mathbf{x}) \gamma(v) \left[\lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta f(t, \mathbf{x}_s, v) ds \right] dx dt dv. \end{aligned}$$

In the last equality we have used that $\mathbf{v}(t, \cdot) \in \mathbf{V}^1(\Omega)$ for a. a. $t \in [0, T]$ with Theorems 6.5.3, 6.5.4 of [13]; $-\nabla d \equiv \mathbf{n}$ on Γ with $\Gamma \in C^2$ and also the dominated convergence theorem applied for the bounded sequence

$$0 \leq \frac{1}{\delta} \int_0^\delta f(t, \mathbf{x}_s, v) ds \leq 1 \quad \text{for a. a. } (t, \mathbf{x}, v) \in \Gamma_T \times \mathbb{R},$$

where $g'(v)\mathbf{b}_n \neq 0$. Denoting by $f^b := \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \int_0^\delta f(t, \mathbf{x}_s, v) ds$, we have $\mathbf{f}_\gamma \cdot \mathbf{n}|_{\Gamma_T} \equiv g'(v)\mathbf{b}_n \langle f^b, \gamma \rangle$. Since γ is an arbitrary function, hence we can denote $\mathbf{f} \cdot \mathbf{n}|_{\Gamma_T} = g'(v)\mathbf{b}_n f^b$. The function f^b fulfills

$$0 \leq f^b \leq 1 \quad \text{on } \Gamma_T \times \mathbb{R} \quad \text{and} \quad \partial_v f^b \leq 0 \quad \text{in } \mathcal{D}'(\Gamma_T \times \mathbb{R}). \quad (5.54)$$

2.2. Let us take a positive function $\psi \in C_0^\infty((0, T) \times \mathbb{R}^d)$. If we set $\phi(t, \mathbf{x}) = (1 - \zeta_\delta(d(\mathbf{x}))) \psi(t, \mathbf{x})$ in (5.47), (5.48) with ζ_δ defined in (5.37) and pass to the limit as $\delta \rightarrow 0^+$, we obtain respectively

$$\int_{\Gamma_T} \psi \left[\int_v^1 g'(s) \mathbf{b}_n f^b(t, \mathbf{x}, s) ds + M |u_b - v|^+ \right] dx dt \geq 0$$

and

$$\int_{\Gamma_T} \psi \left[\int_0^v g'(s) \mathbf{b}_n (1 - f^b(t, \mathbf{x}, s)) ds + M |u_b - v|^- \right] dx dt \geq 0.$$

Hence defining the functions

$$\begin{aligned} m_+^b &:= \int_v^1 g'(s) \mathbf{b}_n f^b(t, \mathbf{x}, s) ds + M |u_b - v|^+, \\ m_-^b &:= \int_0^v g'(s) \mathbf{b}_n (1 - f^b(t, \mathbf{x}, s)) ds + M |u_b - v|^- \end{aligned}$$

for $(t, \mathbf{x}, v) \in \Gamma_T \times \mathbb{R}$, it is not difficult to check that, the positive functions $m_\pm^b(t, \mathbf{x}, v) \in L^2(\Gamma_T; W^{1,\infty}(\mathbb{R}))$ and satisfy

$$\begin{aligned} g'(v) \mathbf{b}_n f^b &= -M \operatorname{sgn}^+(u_b - v) - \partial_v m_+^b \quad \text{and} \\ m_+^b &= 0 \quad \text{for } v \geq 1; \\ g'(v) \mathbf{b}_n (1 - f^b) &= -M \operatorname{sgn}^-(u_b - v) + \partial_v m_-^b \quad \text{and} \\ m_-^b &= 0 \quad \text{for } v \leq 0. \end{aligned} \tag{5.55}$$

Due to (5.54)-(5.55) we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} |g'(v) \mathbf{b}_n|^- f^b(v) (1 - f^b(v)) dv = \int_0^{u_b} f^b \{M \operatorname{sgn}^-(u_b - v) - \partial_v m_-^b\} dv \\ &+ \int_{u_b}^1 \{M \operatorname{sgn}^+(u_b - v) + \partial_v m_+^b\} (1 - f^b) dv = -(f^b m_-^b)|_{v=u_b-0} \\ &+ \int_0^{u_b} \partial_v f^b m_-^b dv - m_+^b (1 - f^b)|_{v=u_b+0} + \int_{u_b}^1 m_+^b \partial_v f^b dv \leq 0 \quad \text{a.e. on } \Gamma_T. \end{aligned}$$

A formal integration on v by parts in the last identity can be justified by mollifying the function f^b and taking the limit transition on a mollifying parameter. Therefore f^b satisfies (5.53). ■

Lemma 5.4. *We have*

$$f = f^2 \quad \text{a. e. in } \Omega_T \times \mathbb{R}. \tag{5.56}$$

Proof. The equations (5.51) are written as

$$\begin{aligned} \partial_t f + \operatorname{div}_{\mathbf{x}}(g'(v) \mathbf{v} f) &= \partial_v m_+ \\ \partial_t (1 - f) + \operatorname{div}_{\mathbf{x}}(g'(v) \mathbf{v} (1 - f)) &= -\partial_v m_- \quad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \end{aligned} \tag{5.57}$$

We have that $\mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$ and $g'(v)$ is a constant with respect of the variables (t, \mathbf{x}) , such that $g' \in L^\infty(\mathbb{R})$ by Remark 5.2. Hence we can apply the renormalization theorem to the left parts of (5.57) (see for instance Theorem 4.3 in [8]) and get, that the function $F := f(1 - f)$ satisfies

$$F_t + \operatorname{div}_{\mathbf{x}}(g'(v) \mathbf{v} F) \leq 0 \quad \text{in } \mathcal{D}'(\mathbb{R} \times \Omega_T). \tag{5.58}$$

It means that the equations in (5.57) are regularized on a parameter θ , multiplied by $(1 - f^\theta)$ and f^θ , respectively (f^θ being the regularization of f) and finally taken the limit on $\theta \rightarrow 0$. The inequality in (5.58) follows from the relation $\int_{\mathbb{R}} \frac{\partial m_{\pm}^\theta}{\partial v} (1 - f^\theta) - \frac{\partial m_{\pm}^\theta}{\partial v} f^\theta dv = \int_{\mathbb{R}} (m_{+}^\theta + m_{-}^\theta) \frac{\partial f^\theta}{\partial v} dv \leq 0$ in view of (5.49) and (5.50).

Now let us define

$$\psi := (\zeta_\varepsilon(v + \varepsilon^{-1}) - \zeta_\varepsilon(v - \varepsilon^{-1})) \psi_1^\delta(t) \psi_2^\delta(\mathbf{x})$$

with $\psi_1^\delta(t) := (\zeta_\delta(t) - \zeta_\delta(t - t_0 + \delta))$ for $t_0 \in (2\delta, T)$ and $\psi_2^\delta(\mathbf{x}) := \zeta_\delta(d(\mathbf{x}))$ for $\mathbf{x} \in \Omega$. Choosing $\psi = \psi(v, t, \mathbf{x})$ as a test function in the respective integral form of (5.58) and taking the limit transition on $\varepsilon \rightarrow 0$, with the help of (5.49) and (5.50), we get the inequality

$$\begin{aligned} & \frac{1}{\delta} \int_{t_0-\delta}^{t_0} \int_{\Omega \times \mathbb{R}} F \psi_2^\delta(\mathbf{x}) dt dv d\mathbf{x} \leq \frac{1}{\delta} \int_0^\delta \int_{\Omega \times \mathbb{R}} F \psi_2^\delta(\mathbf{x}) dt dv d\mathbf{x} \\ & + \frac{1}{\delta} \int_{0 \leq d(\mathbf{x}) \leq \delta} \int_{\mathbb{R} \times [0, T]} |g'(v)(\mathbf{v} \cdot \nabla d)|^- F \psi_1^\delta(t) dt dv d\mathbf{x} \\ = & : C_1^\delta + C_2^\delta. \end{aligned} \quad (5.59)$$

Due to the following simple inequality

$$-\frac{1}{\delta} \int_0^\delta z^2(s) ds \leq -\left(\frac{1}{\delta} \int_0^\delta z(s) ds\right)^2, \quad (5.60)$$

which is valid for any positive integrable function $z = z(s)$, we have that

$$C_1^\delta \leq \int_{\Omega \times \mathbb{R}} [f^\delta - (f^\delta)^2] \psi_2^\delta(\mathbf{x}) dv d\mathbf{x},$$

where $f^\delta := \frac{1}{\delta} \int_0^\delta f(t) dt$. Since $0 \leq f^\delta \leq 1$, in view of the dominated convergence theorem and Proposition 5.3, we derive

$$\limsup_{\delta \rightarrow 0} C_1^\delta \leq \int_{\Omega \times \mathbb{R}} (f^0 - f^{02}) dv d\mathbf{x} = 0.$$

Let us now consider the term C_2^δ . Since $\Gamma \in C^2$, there exists a small δ , such that any point $\mathbf{x} \in S_\delta := \{\mathbf{x} \in \Omega : d(\mathbf{x}) < \delta\}$ has a unique projection $\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x})$ on the boundary Γ . In the set S_δ , we have that $\nabla d(\mathbf{x}) = -\mathbf{n}(\mathbf{x}_0) + O(\delta)$ and the Jacobian of the change of variables $\mathbf{x} \leftrightarrow (\mathbf{x}_0, s)$ with $s := d(\mathbf{x})$ is equal to $\frac{D(\mathbf{x})}{D(\mathbf{x}_0, s)} = 1 + O(\delta)$, since (\mathbf{x}_0, s) forms the orthogonal coordinate system at $s = 0$. In view of $\mathbf{v}(t, \cdot) \in \mathbf{V}^1(\Omega)$ for a. a. $t \in (0, T)$, we can apply theorems 6.5.3-6.5.4 of [13] and obtain with the help of (5.60) the following inequality

$$C_2^\delta \leq \int_{\Gamma_T \times \mathbb{R}} |g'(v) \mathbf{b}_n(t, \mathbf{x}_0)|^- [f^\delta - (f^\delta)^2] dv dt d\mathbf{x}_0 + O(\delta^\alpha).$$

Here $f^\delta := \frac{1}{\delta} \int_0^\delta f(\cdot, \cdot, \mathbf{x}) ds$. Hence Proposition 5.3 implies

$$\limsup_{\delta \rightarrow 0} C_2^\delta \leq \int_{\Gamma_T \times \mathbb{R}} |g'(v) \mathbf{b}_n(t, \mathbf{x})|^- (f^b - (f^b)^2) dv dt d\mathbf{x} = 0.$$

Finally integrating (5.59) over $t_0 \in [2\delta, T]$, applying Fubini's theorem to the left part of the inequality and taking the limit on $\delta \rightarrow 0$, we get $\int_{\Omega_T \times \mathbb{R}} F dv dt d\mathbf{x} \leq 0$. Therefore $F \equiv 0$ a.e. in $\Omega_T \times \mathbb{R}$. \blacksquare

Since f is monotone decreasing on v and f takes only the values 0 and 1, a. e. in $\Omega_T \times \mathbb{R}$, there exists a function $z = z(t, \mathbf{x})$, such that

$$f(t, \mathbf{x}, v) = \text{sgn}^+(z(t, \mathbf{x}) - v).$$

Therefore for any $G \in C^1([0, 1])$, $G(0) = 0$

$$G(u^\varepsilon) = \int_0^1 G'(v) f^\varepsilon(\cdot, \cdot, v) dv \rightharpoonup \int_0^1 G'(v) f(\cdot, \cdot, v) dv = G(z)$$

weakly $*$ in $L^\infty(\Omega_T)$. This implies $z = u$ and the strong convergence of $\{u^\varepsilon\}$ to u in $L^p(\Omega_T)$ for any $p < \infty$. Therefore, the function \mathbf{v} fulfills the integral identity (4.22). And if we take the sum of the (in)equalities (5.47), (5.48), we derive that u satisfies (4.21), that ends the proof of Theorem 4.3.

Remark 5.5. *Let us note that the measures m_+, m_- and the limit functions u, \mathbf{v} with $f(t, \mathbf{x}, v) = \text{sgn}^+(u(t, \mathbf{x}) - v)$ satisfy all relations (5.47)-(5.51) too.*

6 Statement of the quasi-stationary Stokes B - L system

For a given viscous parameter $\nu > 0$, we consider the following initial-boundary value problem, denoted as $\mathbf{IBVP}_{\tau=0}$:

Find a pair $(u, \mathbf{v}) = (u(t, \mathbf{x}), \mathbf{v}(t, \mathbf{x})) : \Omega_T \rightarrow \mathbb{R} \times \mathbb{R}^d$ solution to the quasi-stationary Stokes-Buckley-Leverett system in the domain Ω_T

$$\partial_t u + \text{div}(\mathbf{v} g(u)) = 0, \quad (6.61)$$

$$-\nu \Delta \mathbf{v} + h(u) \mathbf{v} = -\nabla p, \quad \text{div}(\mathbf{v}) = 0, \quad (6.62)$$

satisfying the boundary conditions

$$(u, \mathbf{v}) = (u_b, \mathbf{b}) \quad \text{on } \Gamma_T, \quad (6.63)$$

and the initial condition

$$u = u_0 \quad \text{in } \Omega. \quad (6.64)$$

We assume that our data $g, h, u_b, u_0, \mathbf{b}$ satisfy the following regularity properties

$$\begin{aligned} g, h &\in W_{\text{loc}}^{1,\infty}(\mathbb{R}) \quad \text{with } 0 < h_0 \leq h(u), \\ 0 &\leq u_b \leq 1 \quad \text{on } \Gamma_T, \\ 0 &\leq u_0 \leq 1 \quad \text{in } \Omega, \end{aligned} \tag{6.65}$$

$$\mathbf{b} \in \mathbf{G}(\Gamma_T). \tag{6.66}$$

Definition 6.1. *A pair of functions*

$$(u, \mathbf{v}) \in L^\infty(\Omega_T) \times L^2(0, T; \mathbf{V}^1(\Omega))$$

is called a weak solution to the $\mathbf{IBVP}_{\tau=0}$: (6.61)-(6.64), if the pair (u, \mathbf{v}) satisfies the integral inequality

$$\begin{aligned} &\iint_{\Omega_T} (|u - v| \phi_t + \text{sgn}(u - v)(g(u) - g(v)) \mathbf{v} \cdot \nabla \phi) \, d\mathbf{x} \, dt \\ &+ \int_{\Gamma_T} M |u_b - v| \phi \, d\mathbf{x} \, dt + \int_{\Omega} |u_0 - v| \phi(0, x) \, d\mathbf{x} \geq 0, \end{aligned} \tag{6.67}$$

for any fixed $v \in \mathbb{R}$, where $M := K|\mathbf{b}_n|$ on Γ_T with $K := \|g'\|_{L^\infty(\mathbb{R})}$ and for any nonnegative function $\phi \in C_0^\infty((-\infty, T) \times \mathbb{R}^d)$ and also the following integral identity

$$\int_{\Omega} [\nu \nabla \mathbf{v} : \nabla \boldsymbol{\psi} + h(u) \mathbf{v} \cdot \boldsymbol{\psi}] \, d\mathbf{x} = 0 \quad \text{for a. a. } t \in (0, T) \tag{6.68}$$

holds for any $\boldsymbol{\psi} \in \mathbf{C}_0^1(\Omega)$. Moreover the trace of \mathbf{v} is equal to \mathbf{b} on Γ_T .

Theorem 6.2. *If the data $g, h, u_b, u_0, \mathbf{b}$ fulfills the regularity properties (6.65)-(6.66), then the $\mathbf{IBVP}_{\tau=0}$ has a weak solution (u, \mathbf{v}) , satisfying*

$$\begin{aligned} 0 &\leq u \leq 1 \quad \text{a. e. in } \Omega_T, \\ \mathbf{v}, \partial_t \mathbf{v} &\in L^2(0, T; \mathbf{V}^1(\Omega)). \end{aligned}$$

6.1 Existence of weak solution. The limit transition on $\tau \rightarrow 0$

Let us choose some function $\mathbf{v}_0 \in \mathbf{V}^0(\Omega)$, such that

$$\mathbf{v}_0 \cdot \mathbf{n} = \mathbf{b}(0) \cdot \mathbf{n} \quad \text{in } H^{-1/2}(\Gamma).$$

Then, due to Theorem 4.3, for each $\tau > 0$, there exists a solution $(u^\tau, \mathbf{v}^\tau)$ for the problem \mathbf{IBVP}_τ : (4.12)-(4.15), satisfying (4.23). Hereupon the issue is to pass to the limit on the parameter $\tau \rightarrow 0$ and, as a consequence, to derive the solvability of $\mathbf{IBVP}_{\tau=0}$.

Proposition 6.3. *There exists a pair $(u, \mathbf{v}) \in L^\infty(\Omega_T) \times L^2(0, T; \mathbf{V}^1(\Omega))$, with $\partial_t \mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$, and a subsequence of $\{u^\tau, \mathbf{v}^\tau\}_{\tau>0}$, such that*

$$u^\tau \rightharpoonup u \quad * \text{-weakly in } L^\infty(\Omega_T), \quad (6.69)$$

$$\mathbf{v}^\tau \rightarrow \mathbf{v} \quad \text{strongly in } L^2(\Omega_T). \quad (6.70)$$

Proof. The convergence (6.69) follows from the first estimate of (4.23). Hence it remains to show (6.70).

For each $\tau > 0$, let us consider the quasi-stationary Stokes type system

$$\begin{cases} -\nu \Delta \mathbf{B}^\tau + h(u^\tau) \mathbf{B}^\tau = -\nabla \pi^\tau, & \operatorname{div}(\mathbf{B}^\tau) = 0 \quad \text{in } \Omega_T, \\ \mathbf{B}^\tau = \mathbf{b} \quad \text{on } \Gamma_T. \end{cases}$$

The function $\mathbf{z}^\tau := \mathbf{B}^\tau - \mathbf{v}_b$, where \mathbf{v}_b is the solution of (5.38), fulfills the system

$$\begin{cases} -\nu \Delta \mathbf{z}^\tau + h(u^\tau) \mathbf{z}^\tau = -\nabla(\pi^\tau - p_b) + \mathbf{f}^\tau, & \operatorname{div}(\mathbf{z}^\tau) = 0 \quad \text{in } \Omega_T, \\ \mathbf{z}^\tau = \mathbf{0} \quad \text{on } \Gamma_T, \end{cases}$$

with $\mathbf{f}^\tau := -h(u^\tau) \mathbf{v}_b$. Therefore, for a.a. $t \in (0, T)$, $\mathbf{z}^\tau(t)$ satisfies the following estimate

$$\nu \|\nabla \mathbf{z}^\tau\|_{\mathbf{L}^2(\Omega)}^2 \leq \int_{\Omega} |(\mathbf{f}^\tau \cdot \mathbf{z}^\tau)| \, dx \leq \frac{\nu}{2} \|\nabla \mathbf{z}^\tau\|_{\mathbf{L}^2(\Omega)}^2 + C \|\mathbf{v}_b\|_{\mathbf{L}^2(\Omega)}^2. \quad (6.71)$$

Now, for a.a. $t_0, t_1 \in (0, T)$, we could write

$$\|\mathbf{v}_b(t_1, \cdot)\|^2 - \|\mathbf{v}_b(t_0, \cdot)\|^2 = \int_{t_0}^{t_1} \left(\partial_t \int_{\Omega} \mathbf{v}_b^2 \, dx \right) dt =: J,$$

hence by (5.39)

$$|J| \leq \|\mathbf{v}_b\|_{\mathbf{L}^2(\Omega_T)}^2 + \|\partial_t \mathbf{v}_b\|_{\mathbf{L}^2(\Omega_T)}^2 \leq C$$

and we have that $\mathbf{v}_b \in C([0, T]; \mathbf{V}^0(\Omega_T))$. Consequently, by (6.71), it follows that

$$\|\mathbf{B}^\tau\|_{L^\infty(0, T; \mathbf{V}^1(\Omega))} \leq C. \quad (6.72)$$

Here and below, C are denoted constants, which could change from one to another statement, being independent of the parameter τ .

Since the function u^τ is the solution of (4.12) (in the weak form), the pair $\mathbf{Z}^\tau := \partial_t \mathbf{z}^\tau$ fulfills the system

$$\begin{cases} -\nu \Delta \mathbf{Z}^\tau + h(u^\tau) \mathbf{Z}^\tau = -\nabla Q^\tau + \mathbf{R}^\tau, & \operatorname{div}(\mathbf{Z}^\tau) = 0 \quad \text{in } \Omega_T, \\ \mathbf{Z}^\tau = \mathbf{0} \quad \text{on } \Gamma_T, \end{cases} \quad (6.73)$$

with $Q^\tau := \partial_t(\pi^\tau - p_b)$ and \mathbf{R}^τ is given by

$$\mathbf{R}^\tau := \operatorname{div} \left(r(u^\tau) (\mathbf{B}^\tau \otimes \mathbf{v}^\tau) \right) - r(u^\tau) (\nabla \mathbf{B}^\tau) \mathbf{v}^\tau - h(u^\tau) \partial_t \mathbf{v}_b,$$

where $r(u) := \int_0^u h'(s)g'(s) ds$. From (6.73), we obtain

$$\begin{aligned} \nu \|\nabla \mathbf{Z}^\tau\|_{\mathbf{L}^2(\Omega)}^2 &\leq \int_{\Omega} (\mathbf{R}^\tau \cdot \mathbf{Z}^\tau) d\mathbf{x} \leq C \|\nabla \mathbf{Z}^\tau\|_{\mathbf{L}^2(\Omega)} \\ &\quad \times \left\{ \|\mathbf{v}^\tau\|_{\mathbf{L}^4(\Omega)} \left[\|\mathbf{B}^\tau\|_{\mathbf{L}^4(\Omega)} + \|\nabla \mathbf{B}^\tau\|_{\mathbf{L}^2(\Omega)} \right] + \|\partial_t \mathbf{v}_b\|_{\mathbf{L}^2(\Omega)} \right\}. \end{aligned}$$

The embedding theorem $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and (4.23), (5.39), (6.72) imply

$$\|\partial_t \mathbf{B}^\tau\|_{L^2(0,T;\mathbf{H}^1(\Omega))} \leq C. \quad (6.74)$$

Finally we consider the difference $\mathbf{w}^\tau := \mathbf{v}^\tau - \mathbf{B}^\tau$, which satisfies the system

$$\begin{aligned} \tau \partial_t \mathbf{w}^\tau - \nu \Delta \mathbf{w}^\tau + h(u^\tau) \mathbf{w}^\tau &= -\nabla(p^\tau - \pi^\tau) + \tau \mathbf{f}^\tau, \quad \operatorname{div}(\mathbf{w}^\tau) = 0, \\ \mathbf{w}^\tau|_{\Gamma_T} &= 0, \quad \mathbf{w}^\tau|_{t=0} = \mathbf{v}_0 - \mathbf{B}^\tau|_{t=0}, \end{aligned}$$

with $\mathbf{f}^\tau := -\partial_t \mathbf{B}^\tau$. If we multiply the first equation of this system by \mathbf{w}^τ and integrate over Ω , we obtain

$$\begin{aligned} \frac{d}{dt} \left(\frac{\tau}{2} \|\mathbf{w}^\tau\|_{\mathbf{L}^2(\Omega)}^2 \right) + \nu \|\nabla \mathbf{w}^\tau\|_{\mathbf{L}^2(\Omega)}^2 &\leq C \int_{\Omega} |(\tau \mathbf{f}^\tau \cdot \mathbf{w}^\tau)| d\mathbf{x} \\ &\leq \frac{\nu}{2} \|\nabla \mathbf{w}^\tau\|_{\mathbf{L}^2(\Omega)}^2 + C\tau^2 \|\mathbf{f}^\tau\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned}$$

Integrating the last inequality over the time interval $(0, t)$ and using (6.66), (6.74), we deduce

$$\|\mathbf{v}^\tau - \mathbf{B}^\tau\|_{L^2(0,T;\mathbf{V}^1(\Omega))}^2 \leq C\tau. \quad (6.75)$$

Obviously the derived estimates (6.72), (6.74), (6.75) imply the existence of a function $\mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$, satisfying the strong convergence (6.70) for some subsequence of $\{\mathbf{v}^\tau\}_{\tau>0}$. ■

Of course, the convergence (6.69)-(6.70) is not sufficient to take the limit transition on $\tau \rightarrow 0$ in the system (6.61)–(6.62), since we need the strong convergence of a subsequence for $\{u^\tau\}_{\tau>0}$. To get this strong convergence, we can apply the Kinetic approach, developed in Section 5 and prove Theorem 6.2. In fact, we have to repeat all considerations of the section 5.2.3 (see also Remark 5.5), considering the parameter τ , instead of ε in (5.40)–(5.45) (without viscous terms depending on ε).

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References

- [1] ANTONTSEV S.N., KAZIKHOV A.V., MONAKHOV V.N., *Boundary-value problems in mechanics of non-homogeneous fluids*. Studies in Math. and its Appl., Vol. 22, North-Holland, 1990.
- [2] ARBOGAST T., *The existence of weak solutions to single porosity and simple dual-porosity models of two-phase incompressible flow*, Nonlinear Anal., **19** (11) (1992), 1009-1031.
- [3] CATTABRIGA L., *Su un problema al contorno relativo al sistema di equazioni di Stokes (Italian)*, Rend. Sem. Mat. Univ. Padova 31, (1961), 308–340.
- [4] CHEN G.-Q., FRID H., *Divergence measure fields and hyperbolic conservation laws*. Arch. Rational Mech. Anal. 147 (1999) 89–118.
- [5] CHEN Z., *Degenerate Two-Phase Incompressible Flow: I. Existence, Uniqueness and Regularity of a Weak Solution*, J. Dif. Equations, **171**, Issue 2, (2001), 203-232.
- [6] CÓRDOBA D., GANCEDO F., ORIVE R. *Analytical behavior of two-dimensional incompressible flow in porous media. J. Math. Physics*, **48**(6) 065206 (2007); doi:10.1063/1.2404593 (19 pages).
- [7] DAFERMOS C.M., *Hyperbolic conservation laws in continuum physics*, 2nd edition. Springer Verlag, 2005.
- [8] DE LELLIS C., *Ordinary differential equations with rough coefficients and the renormalization theorem of Ambrosio*, Bourbaki Seminar, Preprint, (2007) 1-26.
- [9] DIPERNA R.J., LIONS P.L., *Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math.* **98**, (1989) 511–547.
- [10] FRID H., *Solution to the Initial Boundary-Value Problem for the Regularized Buckley-Leverett System*, Acta Applicandae Mathematicae, 38, 239–265 (1995).
- [11] FARWIG R., GALDI G.P., SOHR H., *A New Class of Weak Solutions of the Navier–Stokes Equations with Nonhomogeneous Data*, J. Math. Fluid Mech., **8** (2006) 423–444.
- [12] HORNING U., *Homogenization and Porous Media*, Interdisciplinary Applied Math., Vol. 6, Springer, (1996).
- [13] KUFNER A., JONH O., FUČIK S., *Function Spaces*. Noordhoff Intern. Publishing, Leyden (1977).
- [14] LADYZHENSKAYA O.A., *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, New York-London, 1969.

- [15] LADYZHENSKAYA O.A., SOLONNIKOV V.A., URAL'TSEVA N.N., *Linear and quasilinear equations of parabolic type*. American Mathematical Society, Providence RJ (1968).
- [16] LENZINGER M., SCHWEIZER B., *Two-phase flow equations with outflow boundary conditions in the hydrophobic hydrophilic case*, Nonlinear Analysis: Theory, Methods & Applications, **73**, Issue 4 (2010), 840-853.
- [17] LIONS P.-L., PERTHAME B., TADMOR E., *Kinetic formulation for isentropic gas dynamics and p-systems*. Comm. Math. Phys. 163 (1994), 415–431.
- [18] LIONS P.-L., PERTHAME B., TADMOR E., *A kinetic formulation of multidimensional scalar conservation laws and related equations*. J. AMS 7 (1994), 169–191.
- [19] LUCKHAUS S., PLOTNIKOV P.I., *Entropy solutions to the Buckley-Leverett equations*. Siberian Mathematical Journal 41, N. 2 (2000), 169–191.
- [20] MALEK J., NECAS J., ROKYTA M., RUZICKA M., *Weak and measure-valued solutions to evolutionary PDEs*. Chapman&Hall, London (1996).
- [21] NEVES W., *Scalar multidimensional conservation laws IBVP in noncylindrical Lipschitz domains*, Journal of Diff. Equations 192 (2003) 360–395.
- [22] OTTO F., *Initial-boundary value problem for a scalar conservation law*, C.R. Acad. Sci. Paris 322 (1996) 729–734.
- [23] PANOV E. YU., *Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux*, Arch. Ration. Mech. Anal., **195**, no 2, 643–673 (2010).
- [24] PEREPETLITSA I., SHELUKHIN V., *On Global Solutions of a Boundary-Value Problem for the one-dimensional Buckley-Leverett Equations*, Applicable Analysis, **73**, no 3–4, 325–343 (1999).
- [25] PERTHAME B., DALIBARD A.-L., *Existence of solutions of the hyperbolic Keller-Segel model*, Trans. Amer. Math. Soc., **361**, 2319-2335 (2009).
- [26] PERTHAME B., *Kinetic formulation of conservation laws*, Oxford University Press, 2002.
- [27] SAZHENKOV S. A., *Entropy solutions to the Verigin ultraparabolic problem*, Siberian Mathematical Journal 49, No. 2, 362–374 (2008).
- [28] SCHEIDEGGER A.E., *Hydrodynamics in Porous Media*, Handbuch der Physik Vol. VIII/2, Flügge, Springer, (1963).
- [29] SCHEIDEGGER A.E., *The Physics of Flow Through Porous Media*, 3rd ed, University of Toronto Press, Toronto (1974).

- [30] SHEU L.-J., *An autonomous system for chaotic convection in a porous medium using a thermal non-equilibrium model*, Chaos, Solitons and Fractals, **30** (2006) 672–689.
- [31] STRAUGHAN B., *Stability and Wave Motion in porous media*, Applied Math. Sciences Vol. 165, Springer, (2008).
- [32] TEMAM R., *Navier-Stokes equations*, Theory and numerical analysis. AMS Chelsea publishing, Providence, Rhode Island (2001).
- [33] WANG B., LIN S., *Existence of global attractors for the three-dimensional Brinkman Forchheimer equation*, Math. Meth. Appl. Sci., **31** (2008), 1479–1495.